

## ON THE HORSESHOE CONJECTURE FOR MAXIMAL DISTANCE MINIMIZERS

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**ABSTRACT.** We study the properties of sets  $\Sigma$  having the minimal length (one-dimensional Hausdorff measure) over the class of closed connected sets  $\Sigma \subset \mathbb{R}^2$  satisfying the inequality  $\max_{y \in M} \text{dist}(y, \Sigma) \leq r$  for a given compact set  $M \subset \mathbb{R}^2$  and some given  $r > 0$ . Such sets can be considered shortest possible pipelines arriving at a distance at most  $r$  to every point of  $M$  which in this case is considered as the set of customers of the pipeline.

We prove the conjecture of Miranda, Paolini and Stepanov about the set of minimizers for  $M$  a circumference of radius  $R > 0$  for the case when  $r < R/4.98$ . Moreover we show that when  $M$  is a boundary of a smooth convex set with minimal radius of curvature  $R$ , then every minimizer  $\Sigma$  has similar structure for  $r < R/5$ . Additionally we prove a similar statement for local minimizers.

**KEYWORDS.** Steiner tree, locally minimal network, maximal distance minimizer.

**AMS SUBJECT CLASSIFICATION:** 49Q10, 49Q20, 49K30; 90B10, 90C27.

## 1. INTRODUCTION

For a given compact set  $M \subset \mathbb{R}^2$  consider the functional

$$F_M(\Sigma) := \max_{y \in M} \text{dist}(y, \Sigma),$$

where  $\Sigma$  is a subset of  $\mathbb{R}^2$  and  $\text{dist}(y, \Sigma)$  stands for the euclidian distance between  $y$  and  $\Sigma$ . The quantity  $F_M(\Sigma)$  will be further called *energy* of  $\Sigma$ . Consider the class of closed connected sets  $\Sigma \subset \mathbb{R}^2$  satisfying  $F_M(\Sigma) \leq r$  for some  $r > 0$ . We are interested in properties of sets of the minimal length (one-dimensional Hausdorff measure)  $\mathcal{H}^1(\Sigma)$ . Such sets will be further simply referred to as *minimizers*. They can be viewed as shortest possible pipelines arriving at a distance at most  $r$  to every point of  $M$  which in this case is considered as the set of customers of the pipeline.

In [10] it is proven (even in the general  $n$ -dimensional case  $M \subset \mathbb{R}^n$ ) that the set  $OPT_\infty^*(M)$  of minimizers (for all  $r > 0$ ) is nonempty and coincides with the set  $OPT_\infty(M)$  of solutions of the dual problem: to minimize  $F_M$  over all compact connected sets  $\Sigma \subset \mathbb{R}^2$  with prescribed bound on the total length  $\mathcal{H}^1(\Sigma) \leq l$  (for the corresponding  $l > 0$ ). The latter minimizing problem is quite similar to many other problems of minimizing other functionals over closed connected sets, for instance the average distance with respect to some finite Borel measure (see [3], [5], [6], [9] and [8]) or similar urban planning problems (see [4]). If one minimizes minimum or average distance functional over discrete sets with an priori restriction on the number of connected components (rather over connected one-dimensional sets) one gets another class of closely related problems called  $k$ -center problem and  $k$ -median problem (see e.g. [12], [13], [7] as well as [2, 1] and references therein).

Some basic properties of minimizers to the above introduced problem in the general  $n$ -dimensional case (like the absence of loops and Ahlfors regularity) have been proven in [11]. Further, in [10] the following characterization of minimizers has been studied. To describe it, let from now on  $B_r(x)$  stand for the open ball of radius  $r$  centered at a point  $x$  and  $B_r(M)$  for the open  $r$ -neighborhood of  $M$  i.e.

$$B_r(M) := \bigcup_{x \in M} B_r(x).$$

Consider the following natural notion.

**Definition 1.1.** A point  $x \in \Sigma$  is called *energetic*, if for all  $\rho > 0$  one has

$$F_M(\Sigma \setminus B_\rho(x)) > F_M(\Sigma).$$

Denote the set of all energetic points of  $\Sigma$  by  $G_\Sigma$ .

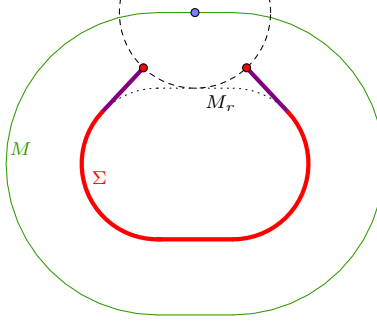


FIGURE 1. A horseshoe.

Let us consider a minimizer  $\Sigma \in OPT_{\infty}^*(M)$  with energy  $r = F_M(\Sigma)$ . Then set  $\Sigma$  can be split into three disjoint sets:

$$\Sigma = E_{\Sigma} \sqcup X_{\Sigma} \sqcup S_{\Sigma},$$

where  $X_{\Sigma} \subset G_{\Sigma}$  is the set of isolated energetic points (i.e. every  $x \in X_{\Sigma}$  is energetic and there is a  $\rho > 0$  possibly depending on  $x$  such that  $B_{\rho}(x) \cap G_{\Sigma} = \{x\}$ ) and  $E_{\Sigma} = G_{\Sigma} \setminus X_{\Sigma}$  is the set of non isolated energetic points. In [10] the following assertions have been proven.

- (a) For every point  $x \in G_{\Sigma}$  there exists an  $y \in M$  such that  $|x - y| = r$  and  $B_r(y) \cap \Sigma = \emptyset$ . If  $X_{\Sigma}$  is not finite, the limit points of  $X_{\Sigma}$  belong to  $E_{\Sigma}$ .
- (b) For all  $x \in S_{\Sigma}$  there exists an  $\varepsilon > 0$  such that  $S_{\Sigma} \cap B_{\varepsilon}(x)$  is either a segment or a regular tripod, i.e. the union of three segments with an endpoint in  $x$  and relative angles of  $2\pi/3$ .
- (c)  $X_{\Sigma}$  is relatively open in  $G_{\Sigma}$ .

We will further use the following notions.

**Definition 1.2.** For a convex closed set  $N$  we define the minimal radius of curvature of its boundary by the formula

$$R(\partial N) := \inf_{x \in \partial N} \sup\{r : B_r(O) \cap \partial N = x \text{ for some } O \in N\}.$$

**Definition 1.3.** For a convex closed set  $N$  we define the inner set  $N_r$  to be the set of all points of  $N$  lying on distance at least  $r$  from the boundary, namely,  $N_r := N \setminus B_r(\partial N)$ .

From now on we define  $N := \text{conv}(M)$ , where  $\text{conv}$  stands for the closed convex envelope, and  $M_r := \partial N_r$ . Note that  $N$ ,  $N_r$ ,  $M$  and  $M_r$  are closed sets. By a *closed convex curve* we mean a *boundary of a convex set*.

Clearly, if  $M$  is a convex closed curve with minimal radius of curvature  $R > r$ , then  $M_r$  is convex closed curve and has minimal radius of curvature at least  $R - r$ .

## 2. MAIN RESULTS

For a given points  $A, B$  we use notations  $[AB]$ ,  $\overline{AB}$  and  $(AB)$  for the corresponding (closed) line segment, ray and line respectively.

We call a *chord* of a closed convex curve  $D$  a line segment connecting two points of  $D$ . A subset of  $D$  is called *arc* of  $D$  if it is a continuous injective image of an interval (it may be a point). Images of the end points of the interval will be called *ends* of the arc; images of internal points of the interval will be called *internal* points of the arc. Whenever there is no confusion the arc with ends  $A, B$  will be denoted by  $\overset{\circ}{AB}$  and its length by  $|\overset{\circ}{AB}|$  (not to be confused with the length of segment connecting  $A$  and  $B$  which is denoted by  $|AB|$ ).

We say that an arc  $q = \overset{\circ}{Q_l Q_r}$  of  $M_r$  is continued by a chord in a set  $\Sigma$  if for some  $i \in \{l, r\}$  there is a chord  $[Q_i X]$  of  $N_r$  such that  $[Q_i X] \subset \Sigma$ .

**Definition 2.1.** Let  $M$  be a closed convex curve with the minimal radius of curvature  $R > r$ . Then the connected curve  $\Sigma$  is called a *horseshoe*, if  $F_M(\Sigma) = r$  and  $\Sigma$  is a union of an arc  $q$  of  $M_r$  with two tangent segments to  $M_r$  in the different ends of  $q$  and ending by energetic points (as shown in Fig. 1).



- (iii) *There exists a function  $r \mapsto a_M(r)$  such that  $a_M(r) \leq 2r$  and the length of any line segment in  $\Sigma_r$  does not exceed  $a_M(r)$ . For the circumference  $M := \partial B_R(O)$  one can take  $a_{\partial B_R(O)}(r) = 2r\sqrt{1 - \frac{r^2}{4R^2}}$ .*

**Remark 2.6.** From the Remark 3.3 it follows that when  $\Sigma \subset N$  is a connected set covering  $M$ , such that  $\mathcal{H}^1(\Sigma) \leq \mathcal{H}^1(\Sigma_{\text{opt}}) + a$ , where  $\Sigma_{\text{opt}}$  is any minimizer, and with  $\Sigma_r$  consisting only of line segments, then the length of any line segment  $\Delta \subset \Sigma_r$  satisfies

$$\mathcal{H}^1(\Delta) \leq 2r + a.$$

A similar statement may be proven for  $M$  being a boundary of a not necessarily convex set, but we restrict the statement to the convex case to avoid excessive technicalities.

**Lemma 2.7.** *Let  $M$  be a closed convex curve with the minimal radius of curvature  $R > 2a_M(r) + r$ , where  $a_M$  is as in Lemma 2.5. Then  $\Sigma_r$  has no Steiner point. Thus  $\Sigma_r$  consists of chords of  $M_r$ .*

Now let us consider an arbitrary connected component  $S$  of  $\Sigma \setminus N_r$  and denote by  $n(S)$  the number of energetic points in  $S$ . Points from  $\bar{S} \cap M_r$  will be further called *entering* points of  $S$ . Denote the number of entering points by  $m(S)$ .

The following lemma says in particular that  $m(S)$  is finite.

**Lemma 2.8.** *Let  $M$  be a closed convex curve with the minimal radius of curvature  $R > 2a_M(r) + r$ . Let  $S$  be a connected component of  $\Sigma \setminus N_r$ . Then  $n(S) \leq 2$ ,  $m(S) \leq 2$ . Further,  $S$  is a locally minimal network connecting the set of entering and energetic points of  $S$ .*

**Remark 2.9.** Let  $\Sigma$  be a local minimizer in the sense of Definition 2.3. Then in view of Remark 2.6 with  $a := (R - 5r)/4$  every line segment in  $\Sigma_r := \Sigma \cap N_r$  has length not exceeding  $a'_M(r) = 2r + (R - 5r)/4$ . Therefore, in Lemmas 2.7 and 2.8 as well as Corollary 2.13 below we may substitute  $a_M$  with  $a'_M$ , which gives that they remain true when  $R > 2a'_M(r) + r = 5r + (R - 5r)/2$ , that is, still when  $R > 5r$ .

Note that  $\bar{B}_r(S) \cap M$  is always a closed arc by Lemma 2.8. We denote it by  $q_S$ .

**Lemma 2.10.** *Under conditions of Theorem 2.2 every arc in  $\Sigma$  is continued by the segments lying on tangent line to  $M_r$ . The same is true if  $\Sigma$  is a local minimizer in the sense of Definition 2.3 for a convex curve  $M$  and  $r < R/5$ .*

**Remark 2.11.** As it will be clear from the proof, for the claim of the above Lemma 2.10 to hold in the case when  $\Sigma$  is a local minimizer, the requirement that  $r < R/5$  may be a bit weakened.

**Lemma 2.12.** *Let  $M$  be a convex closed curve with the minimal radius of curvature  $R$  and  $\Sigma$  be a minimizer with energy  $r < R$ . Then the following assertions hold.*

- (i) *Let  $S_1, S_2$  be connected components of  $\Sigma \setminus N_r$  or arcs of  $\Sigma \cap M_r$ . Then  $q_{S_1}$  and  $q_{S_2}$  have disjoint interiors.*
- (ii)  *$\Sigma$  has no cycles.*

**Corollary 2.13.** *Suppose that  $\Sigma$  has no Steiner point in  $N_r$ . Consider the following abstract graph with the vertices corresponding to connected components of  $\Sigma \setminus N_r$  and to arcs of  $\Sigma \cap M_r$ , the edges connecting the vertices being defined as follows:*

- *if the two vertices are both connected components of  $\Sigma \setminus N_r$ , then there is an edge between them, if they are connected by a chord of  $M_r$ ,*
- *if one vertex is a connected component  $S$  of  $\Sigma \setminus N_r$  and the other is an arc  $C$  of  $\Sigma \cap M_r$ , then there is an edge between them, if  $\bar{S} \cap C \neq \emptyset$ ,*
- *and, finally, if the two vertices are both arcs of  $\Sigma \cap M_r$ , then there is no edge between them.*

*This graph is a tree and, moreover, if  $R > 2a_M(r) + r$  then it is a path.*

**Proof.** Lemma 2.12(ii) gives us the absence of cycles in the graph and Lemma 2.8 stands that degree of each vertex in this graph is at most 2. Connectedness of  $\Sigma$  gives us that the graph is connected. So the graph is a path if  $R > 2a_M(r) + r$ .  $\square$



**Definition 2.15.** Let  $q$  be a curve (not necessarily injective). We say that the winding of  $q$  is the following object:

$$\text{wind}(q) = \int_a^b d \arg(\gamma'(t)),$$

where  $\gamma : [a, b] \rightarrow \mathbb{R}^2$  is some parametrization of  $q$  and  $\arg$  is the continuous branch of the multifunction  $\text{Arg}$ . In our setting it will almost always coincide with the angle between the tangent lines to the ends of  $q$ .

**Definition 2.16.** Let  $S$  be a connected component of  $\Sigma \setminus N_r$ . Then  $\text{wind}(S)$  stands for the winding number of the  $S \cap \partial T$  parametrized in the clockwise order. In particular, if  $S = S_l$  then  $\text{wind}(S)$  stands for the winding number of the corresponding curve  $S \cap \partial T$  parametrized so that it starts at the entering point and ends at  $S'_l$ , and if  $S = S_r$  then  $\text{wind}(S)$  stands for the winding number of  $S \cap \partial T$  parametrized so that it starts at  $S'_r$  and ends at the entering point.

For rays  $[BC)$ ,  $[CD)$  we let  $\angle([BC), [CD))$  stand for the directed angle from  $[BC)$  to  $[CD)$ .

Now we are ready to state the central Lemma.

**Lemma 2.17.** Let  $M$  be a convex closed curve with the minimal radius of curvature  $R$ . Suppose that  $r < R/5$  (for the circumference  $M := \partial B_R(O)$  one can take  $r < R/4.98$ ) and let  $S$  be a connected component of  $\Sigma \setminus N_r$  or an arc of  $M_r$ . Then the following assertions hold.

- If  $S$  is a middle component or an arc of  $M_r$  then  $\text{wind}(q_S) \leq \text{wind}(S)$ . The equality holds if and only if  $S$  is an arc of  $M_r$ .
- If  $S$  is an ending component then for the left and the right components we have

$$\text{wind}(q_S) \leq \text{wind}(S) + \angle([CS'_l], [S'_l A)) + \angle([AS'_l], a),$$

$$\text{wind}(q_S) \leq \text{wind}(S) + \angle([CS'_r], [S'_r A)) + \angle(a, [AS'_r)),$$

where  $a$  stands for the tangent ray to  $M$  in the point  $A$  directed from the left to the right (see Fig. 3, angles  $\angle([AS'_l], a)$ ,  $\angle(a, [AS'_r))$  are marked red) and  $C$  is the branching point if  $S$  is a tripod and the entering point if  $S$  is a segment (we know that these are the only possibilities by Lemma 2.8). The equality holds if and only if  $S$  is a segment of the tangent line to  $M_r$ .

*Remark 2.18.* If in Lemma 2.17 we assume that  $\Sigma$  has no Steiner points in  $N_r$  then it is enough to request the inequality  $r < R/2.9$  (see proof of Lemma 2.17, Case 1a).

Now the proof of Theorem 2.2 is just few lines.

*Proof of Theorem 2.2.* Note that  $\text{wind}(\partial T) = \text{wind}(M) = 2\pi$ . Hence by Lemma 2.17 every global minimizer  $\Sigma$  consists of arcs of  $M_r$  and line segments tangent to  $M_r$ . Thus it has a unique arc of  $M_r$ , and because of absence of loops it also has some ending components. An ending component can not be an arc. Hence any minimizer is a horseshoe.  $\square$

### 3. PROOFS

Clearly  $\Sigma \subset N$  ( $N$  is a uniformly convex set so one can project the part of  $\Sigma$  lying in  $\mathbb{R}^2 \setminus N$  on  $N$  and length of  $\Sigma$  will strictly decrease).

**Lemma 3.1.** Let  $M$  be a convex closed curve with the minimal radius of curvature  $R > r$  and  $B$  be a ball of radius  $r$  centered in a point of  $N$ . If  $B$  touches  $M$  (tangentially to  $M$ ), then  $B \subset N$ .

Further on we assume that by default  $M$  and  $\Sigma$  to be as in Theorem 2.2. Sometimes we will demand weaker conditions.

The following assertion is valid.

**Lemma 3.2.** Let  $M$  be a convex closed curve with the minimal radius of curvature  $R > r$  and  $\Sigma$  be an arbitrary minimizer for  $M$ . Then the set  $E_\Sigma$  of non-discrete energetic points of  $\Sigma$  is a subset of  $M_r$ .

*Proof.* Suppose the contrary. Then there is such a point  $x \in E_\Sigma \setminus M_r$  that  $\text{dist}(x, M) < r - \varepsilon$  for some positive  $\varepsilon$  and a sequence  $\{x_k\}$  of energetic points from  $B_{\varepsilon/2}(x)$  converging to  $x$ . Because of the convexity of  $N$  and the fact that the minimal radius of curvature of  $M$  exceeds  $r$ , one has that each  $\gamma_k := \bar{B}_r(x_k) \cap M$  is connected, so that we can say that each  $x_k$  covers the arc  $\gamma_k$ . All  $\gamma_j$  have a common point. In fact, for  $z \in M$  such that  $\text{dist}(x, z) = \text{dist}(z, M)$  one has

$$\text{dist}(x_j, z) \leq \text{dist}(x, z) + \text{dist}(x_j, x) \leq r - \varepsilon + \varepsilon/2 = r - \varepsilon/2,$$

so that  $z \in \gamma_j$ . One can see that either  $\gamma_i \subset \gamma_j$  or  $\gamma_i \subset \gamma_j \cup \gamma_l$  for some distinct  $i, j, l$ . So one of the points  $x_i, x_j, x_l$  is not energetic which is the desired contradiction.  $\square$

*Proof of Lemma 2.5.* PROOF OF (I): No change in the set  $\Sigma \cap N_r$  influences the value of  $F_M(\Sigma)$ , so that if we take any connected component  $S$  of  $\Sigma \cap N_r$  and substitute it by a Steiner set connecting  $S \cap M_r$  (which must be nonempty because of connectedness of  $\Sigma$  and the requirement  $F_M(\Sigma) \leq r$  which gives  $\Sigma \setminus N_r \neq \emptyset$ ), then the length of the resulting set should remain the same by optimality of  $\Sigma$ , and thus  $S$  is itself a Steiner set connecting  $S \cap \partial N_r$  as claimed.

PROOF OF (II): Recall that  $\Sigma = E_\Sigma \sqcup X_\Sigma \sqcup S_\Sigma$ , where  $X_\Sigma$  is a discrete set of points,  $S_\Sigma$  consists of Steiner trees (hence of line segments) and  $E_\Sigma \subset M_r$  by Lemma 3.2.

PROOF OF (III): Let  $\Sigma'$  be an arbitrary connected component of  $\Sigma \cap N_r$ . Exclude an arbitrary open segment  $\Delta \subset \Sigma'$ . The value of  $F_M$  does not change, i.e.  $F_M(\Sigma') = F_M(\Sigma)$  and  $\Sigma$  splits into two connected components  $\Sigma_1$  and  $\Sigma_2$ , so that  $\Sigma' = \Sigma_1 \sqcup \Sigma_2$ . Obviously  $M \supset B_r(\bar{\Sigma}_1) \cup B_r(\bar{\Sigma}_2)$ . Then by connectedness of  $M$  there is such a point  $A \in M$  that  $A \in \overline{B_r(\bar{\Sigma}_1)} \cap \overline{B_r(\bar{\Sigma}_2)}$ , but then there are points  $B \in \bar{\Sigma}_1$  and  $C \in \bar{\Sigma}_2$  such that  $|AB| \leq r$ ,  $|AC| \leq r$ . Hence the distance between  $\Sigma_1$  and  $\Sigma_2$  does not exceed  $|BC| \leq 2r$  but the length of deleted segment  $\Delta$  does not exceed the distance between the  $\Sigma_1$  and  $\Sigma_2$  in view of optimality of  $\Sigma$  (otherwise one could connect  $\Sigma_1$  with  $\Sigma_2$  shorter). Let  $a_M(r)$  be the maximal value of  $|BC|$  over all the possible choices of  $\Delta$  and (with each choice of  $\Delta$ ) of the points  $B$  and  $C$ .

If  $M = \partial B_R(O)$  the length of the segment  $[BC]$  reaches its maximal value when  $[BC]$  is a chord and  $|AB| = |BC| = r$ . Then we can calculate the maximal value of length of  $[BC]$  in this case:

$$\sin \frac{\angle AOC}{2} = \frac{|AC|}{2|OC|} = \frac{r}{2R},$$

so that

$$|BC| = 2 \sin \angle AOC \cdot |OC| = 4 \sin \frac{\angle AOC}{2} \cos \frac{\angle AOC}{2} |OC| = 2r \sqrt{1 - \frac{r^2}{4R^2}}.$$

$\square$

*Remark 3.3.* To prove Remark 2.6, we substitute the proof of Lemma 2.5(iii) by the following very similar argument. Let  $\Delta \subset \Sigma \cap N_r$  be an arbitrary open segment and  $\Sigma'$  be again the connected component of  $\Sigma \cap N_r$  containing  $\Delta$ . Denote by  $D$  a set (possibly empty) of minimal length such that  $(\Sigma' \setminus \Delta) \cup D$  be closed and connected. Clearly,  $\mathcal{H}^1(D) \leq 2r$ : in fact, if  $\Sigma' \setminus \Delta$  consists of two connected components  $\Sigma_i$ ,  $i = 1, 2$ , recalling that  $\Sigma$  covers  $M$ , we repeat the reasoning of the proof of Lemma 2.5(iii) to show that  $\text{dist}(\Sigma_1, \Sigma_2) \leq 2r$ , which gives the claim. It remains now to observe that

$$\begin{aligned} \mathcal{H}^1(\Sigma \setminus \Delta) + 2r &\geq \mathcal{H}^1(\Sigma \setminus \Delta) + \mathcal{H}^1(D) \geq \mathcal{H}^1((\Sigma \setminus \Delta) \cup D) \\ &\geq \mathcal{H}^1(\Sigma_{\text{opt}}) \geq \mathcal{H}^1(\Sigma) - a, \end{aligned}$$

so that  $\mathcal{H}^1(\Delta) \leq \mathcal{H}^1(\Sigma) - \mathcal{H}^1(\Sigma \setminus \Delta) \leq 2r + a$  concluding the proof of the Remark.

*Proof of Lemma 2.7.* Assume the contrary, i.e. that  $\Sigma$  has a Steiner point  $A_0$  in  $N_r$ . In view of Lemma 3.1 there is a point  $O \in N$  such that  $A_0 \in B_R(O)$  and  $B_R(O) \subset N$  (in particular  $B_{R-r}(O) \subset N_r$ ). Now denote by  $A$  one of the Steiner points of  $\Sigma_r$  nearest to  $O$ , and let  $t := |OA|$ .

Let  $\Sigma'$  stand for the connected component of  $\Sigma \cap N_r$  containing  $A$ . By the structure of a locally minimal network there are three line segments of  $\Sigma'$  started from  $A$ . Consider such a pair of them  $[AA_{-1}], [AA_1]$  that the point  $O$  belongs to the angle  $\angle A_{-1}AA_1$  (not excluding the case it belongs to one of the sides of this angle). Recall that  $\angle A_{-1}AA_1$  is equal to  $2\pi/3$ . Note that by Lemma 2.5 the lengths of  $[AA_{-1}]$  and  $[AA_1]$  do not exceed  $a_M(r)$ . Also note that points  $A_{-1}, A_1$  lie outside of  $B_t(O)$ . Hence either  $[AA_1]$  or

$[AA_{-1}]$  intersects  $B_t(O)$ . We assume without loss of generality that it is  $[AA_1]$ . Denote the intersection of the segment  $[AA_1]$  and the circumference  $\partial B_t(O)$  by  $C$ .

We claim that  $t \leq a_M(r)$ . Supposing the contrary, since  $|AC| \leq |AA_1| \leq a_M(r)$  and  $|OA| = |OC| = t > a_M(r) \geq |AC|$ , we have  $\angle OAC > \pi/3$ , hence the segment  $[AA_{-1}]$  also intersects  $B_t(O)$ . Denote the intersection of the segment  $[AA_{-1}]$  with  $\partial B_t(O)$  by  $D$  and note that  $\angle OAD$  also is greater than  $\pi/3$ , and hence  $\angle CAD > 2\pi/3$  which contradicts the minimality of  $\Sigma$ , showing the claim.

Note that  $A_1, A_{-1}$  belong to  $N_r$  because  $R - r > 2a_M(r) \geq t + a_M(r)$ , and hence  $A_1, A_{-1}$  are Steiner points. Consider a regular hexagon  $P$  with side of length  $a_M(r)$  such that  $A$  is a vertex of  $P$  and the segments  $[AA_1], [AA_{-1}]$  belong to two sides of  $P$ . The following assertions hold.

- $\text{diam } P = 2a_M(r)$ .
- The line segment  $[OA]$  splits the angle  $\angle A_{-1}AA_1 = 2\pi/3$  in two angles, at least one of them is acute. Denote the latter angle by  $\angle OAB$ , where  $B$  is the corresponding vertex of  $P$  (so that  $|AB| = a_M(r)$ ). Then the angle  $\angle OBA$  is also acute because  $|OA| = t \leq a_M(r) = |AB|$ . Therefore the perpendicular from  $O$  to the line  $(AB)$  intersects the latter inside  $[AB]$ , so that  $O$  is inside the square built on  $[AB]$ . But this square is a subset of  $P$  hence  $O \in P$ .
- The above assertions imply that  $P \subset B_{2a_M(r)}(O)$ , and hence  $P \subset N_r$ .

Now let us pick such vertices  $A_{-2}$  and  $A_2$  that  $[A_1A_2], [A_{-1}A_{-2}] \subset \Sigma_r$  and  $O$  belongs to both the angle  $\angle AA_1A_2$  and the angle  $\angle AA_{-1}A_{-2}$ . Clearly  $A_2, A_{-2} \in P \subset N_r$  so they again are Steiner points. Let us define the points  $A_3, A_{-3}$  in the same way:  $[A_2A_3], [A_{-2}A_{-3}] \in \Sigma_r$  and  $O$  belongs to the angles  $\angle A_1A_2A_3$  and  $\angle A_{-1}A_{-2}A_{-3}$ . Points  $A_3, A_{-3}$  also belong to  $P$ , hence to  $N_r$ , hence they also are Steiner points. The six constructed line segments are in the interior of  $N_r$  so there is no terminal point there. Continuing inductively this construction, we arrive at two paths in  $P \subset N_r$ : one path (starting from  $A, A_1, A_2, A_3 \dots$ ) turns left every time and the other one (starting from  $A, A_{-1}, A_{-2}, A_{-3} \dots$ ) turns right every time. Thus  $\Sigma \cap P \subset \Sigma \cap N_r$  contains a cycle or an endpoint, but both cases are impossible for the Steiner tree.  $\square$

*Proof of Lemma 2.8.* Let  $S$  be a connected component of  $\Sigma \setminus N_r$ . First we prove that  $n(S) \leq 2$ . By property (a) of the set of energetic points for every energetic point  $x \in S$  of  $\Sigma$  there is such a point  $y = y(x) \in M$  that  $|xy| = r$  and  $B_r(y) \cap S = \emptyset$ . Then  $y$  can be only the end of the arc  $q_S$ , otherwise  $S = S \setminus B_r(y)$  is not connected. If an end of  $q_S$  corresponds to two different energetic points  $W_1, W_2$  of  $\Sigma \cup S$  then  $q_{W_1} \Delta q_{W_2} = \emptyset$  which is impossible, and hence  $n(S) \leq 2$  as was claimed.

Now let us prove  $m(S) \leq 2$ . Assume the contrary, i.e. the existence of at least three different entering points in  $S$ . Let us denote them sequentially in the clockwise order  $Q_1, Q_2$  and  $Q_3$ . By Lemma 2.5  $\Sigma_r$  consists of chords of  $M_r$ . Denote by  $W$  the end of a chord of  $M_r$  in  $\Sigma_r$  with the other end  $Q_2$ . Then  $|Q_2W| \leq |Q_1W|$  (otherwise we can replace  $[Q_2W]$  by  $[Q_1W]$  in  $\Sigma$  producing the competitor of strictly lower length), and analogously  $|Q_2W| \leq |Q_3W|$ . Note that  $W \notin \bar{S}$  because  $\Sigma_r \setminus (Q_2W)$  would be a competitor of strictly lower length. One can see that points  $Q_1, Q_2, Q_3, W$  belong to  $M_r$  in the above clockwise order. otherwise the arc  $q_{S_W}$  is a subset of  $q_S$ , where  $S_W$  is the connected component of  $\Sigma \setminus N_r$  containing  $W$ , which is impossible.

Hence  $|WQ_2|$  is at least the diameter  $d$  of the maximum ball inscribed in  $N_r$  and touching  $Q_2$  (sometimes it is called *inradius*). Since  $d \geq 2(R - r)$ , we have  $|WQ_2| \geq 2(R - r) > 2r$  contrary to Lemma 2.5(iii), showing this claim.

Finally, note that any subset of  $S$  containing all energetic points of  $S$  covers  $q_S$ , and thus, by substituting  $S$  with the Steiner connection  $\tilde{S}$  of entering and energetic points in  $S$ , we get a connected competitor to  $\Sigma$ , so that by optimality of  $\Sigma$  one would have that the lengths of  $S$  and  $\tilde{S}$  have to be equal, which shows the last claim of the Lemma.  $\square$

*Remark 3.4.* Since by Lemma 2.8  $S$  is a locally minimal network connecting energetic and entering points, then energetic points are endpoints of  $S$ . Suppose  $S$  has two energetic points  $W_1$  and  $W_2$ . Let  $[A_1W_1]$  and  $[A_2W_2]$  be the terminal segments of  $S$  terminating in  $W_1, W_2$  respectively (the case  $A_1 = A_2$  is not excluded). Let  $Q_i \in q_S$  be such points that  $B_r(Q_i) \cap \Sigma = \emptyset$  (by the proof of Lemma 2.8  $Q_i$  are ends of  $q_S$ ),  $i = 1, 2$ . Then each  $Q_i$  belongs to the line  $(A_iW_i)$ ,  $i = 1, 2$ .

*Proof of Remark 3.4.* In fact,  $S$  is a Steiner tree for three or four points with  $A_1, A_2$  branching points. Suppose the contrary i.e.  $A_1, W_1, Q_1$  do not lie on a line. Then one can replace  $[A_1W_1] \cap B_\varepsilon(W_1)$  with  $\varepsilon > 0$



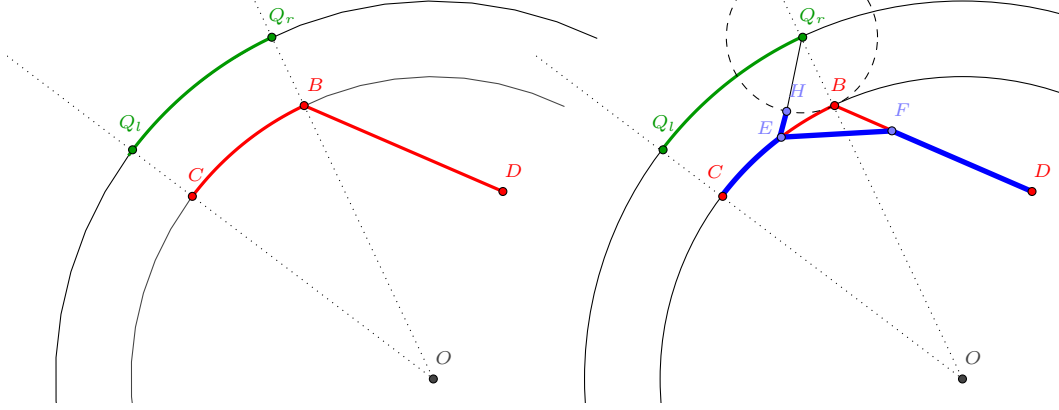


FIGURE 4. Case with an inner segment. Picture to Lemma 2.10

arbitrarily small by the part of the segment  $[DQ_1]$  where  $D = [A_1W_1] \cap \partial B_\varepsilon(W_1)$  producing a competitor of strictly lower length (one easily observes  $q_S$  is still covered by this competitor).  $\square$

*Proof of Lemma 2.10.* Let  $\check{BC} \subset \Sigma$  be an arc of  $M_r$ .

Note that  $\Sigma \setminus M_r$  consists of line segments: when  $\Sigma$  is a minimizer, this follows from Lemmas 2.7 and 2.8, and when  $\Sigma$  is just a local minimizer and  $r < R/5$  it follows from Remark 2.9.

We first show that  $\check{BC}$  cannot contain a branching point (obviously it can be only the end of the arc), say,  $C$ . Suppose the contrary, i.e. that there is both a chord  $[CX] \subset \Sigma \cap N_r$  and a line segment  $[CY] \subset \Sigma \setminus N_r$ . Then the angle between  $M_r$  and  $[CX]$  is at most  $2\pi/3$ : otherwise one can replace  $\Sigma \cap B_\delta(C)$  (for some sufficiently small  $\delta > 0$ ) by the Steiner tree connecting  $(\Sigma \cap \partial B_\delta(C)) \cup \{C\}$  producing a competitor of strictly lower length (note that this reasoning being local, it works both for the case of  $\Sigma$  a true minimizer, as well as for  $\Sigma$  just a local minimizer). Also this angle cannot be less than  $\pi/3$ , otherwise one can replace  $[CD]$  by  $[DD']$  in  $\Sigma$  where  $D := \partial B_\varepsilon(C) \cap [CX]$ ,  $D' := \partial B_\varepsilon(C) \cap \check{BC}$ . Consider the circle  $\omega$  of radius  $R - r$  tangent to  $M_r$  in a point  $C$ . By Lemma 3.1 it lies inside  $N_r$ . Denote by  $Z$  the intersection point of  $[CX]$  and  $\omega$  and by  $O$  the center of  $\omega$ . Since the angle between  $M_r$  and  $[CX]$  is between  $\pi/3$  and  $2\pi/3$ , then  $\angle COZ$  is at most  $\pi - 2|2\angle OCZ - \pi/2| \leq 2\pi/3$ . Therefore

$$|CX| \geq |CZ| = \frac{2(R-r)}{\sin \angle COZ} \geq \frac{2(R-r)}{\sin(2\pi/3)}$$

which gives an immediate contradiction with Lemma 2.5 when  $\Sigma$  is a minimizer, because the right-hand side of the above inequality is strictly greater than  $2r$  for  $r < 2R/5$  (which covers both the general claim for  $r < R/5$  as well as the one for  $M$  being a circumference  $r < R/4.98$ ), so that one has  $|CX| > 2r \geq a_M(r)$ . If  $\Sigma$  is a local minimizer, then the right-hand side of the above inequality for  $r < R/5$  is strictly greater than  $2r + (R - 5r)/4$ , so that one has  $|CX| > 2r + (R - 5r)/4$  contradicting Remark 2.9.

The rest of the proof will use only local arguments, so will work both for the case when  $\Sigma$  is a minimizer or a local minimizer.

First let us prove that neither  $B$  nor  $C$  coincides with  $S'_l$  or  $S'_r$ . Assume the contrary, i.e. that  $B = S'_r$  without loss of generality. Then one can replace  $\check{B}_1B$  by the part of segment  $[B_1A]$ , where  $B_1 = B_\varepsilon(B) \cap \check{BC}$  producing the competitor of strictly lower length. It remains therefore to prove that every arc in  $\Sigma$  is continued by segments of tangent lines to  $M_r$ .

First we prove that an arc cannot be continued by a chord in  $\Sigma$ , then that it is possible to go into the “annulus”  $N \setminus N_r$  from the arc of  $M_r$  only in the tangent direction (which means existence of a line segment  $[BD] \subset \Sigma$ , such that  $D \in N \setminus N_r$  and, moreover,  $(BD)$  is tangent to  $M_r$  at  $B$ ).

We claim that  $\check{BC}$  cannot be continued by a chord in  $\Sigma$ . For a sufficiently small  $\varepsilon > 0$  consider the points  $E \in \check{CB}$ ,  $F \in [BD]$  such that  $|EB| = |BF| = \varepsilon$ . Then  $|EB| = \varepsilon + o(\varepsilon)$  as  $\varepsilon \rightarrow 0^+$ , because  $M_r$  is smooth (the condition on curvature of  $M$  implies  $C^{1,1}$  smoothness). Let  $Q_r \in M$  be such a point that  $|Q_rB| = r$

and  $B_r(Q_r) \cap \Sigma = \emptyset$ , and let  $H$  be the point of intersection of  $[EQ_r]$  and  $\partial B_r(Q_r)$  (see Fig. 4). Then  $(BQ_r)$  is perpendicular to the tangent line to  $M_r$  at the point  $B$ . Thus

$$\begin{aligned} |EH| &= |EQ_r| - |Q_rH| = \sqrt{|EB|^2 + r^2 + o(\varepsilon)} - r = \sqrt{(\varepsilon + o(\varepsilon))^2 + r^2} - r \\ &= r\sqrt{1 + o(\varepsilon)} - r = o(\varepsilon). \end{aligned}$$

Now, since the angle between the arc  $\check{E}B$  and the segment  $[BF]$  is less than  $\pi$ , we get

$$|EF| = \sqrt{2\varepsilon^2 - 2\varepsilon^2 \cos \angle EBF} + o(\varepsilon) = \sqrt{2\varepsilon} \sqrt{1 - \cos \angle EBF} + o(\varepsilon) < 2\varepsilon - |\Omega(\varepsilon)|,$$

and therefore

$$|EH| + |EF| < 2\varepsilon = |\check{E}B| + |BF|$$

for sufficiently small  $\varepsilon > 0$ .

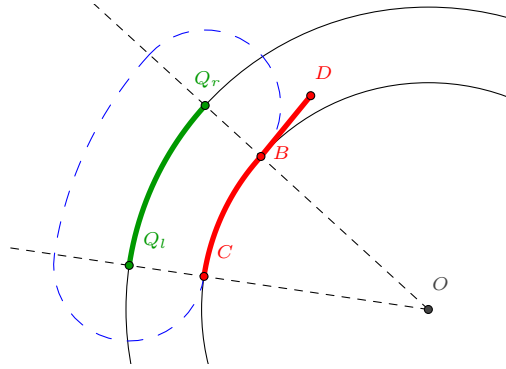


FIGURE 5. Area without  $\Sigma$ -points. Picture to Lemma 2.10.

We claim now that the only way to go inside the “annulus”  $N \setminus N_r$  from the arc of  $M_r$  is in the tangent direction to  $M_r$ , namely, if there is a line segment  $[BD] \subset \Sigma$ , then  $D \in N \setminus N_r$  (this already follows from the above proven claim and from the fact that  $M_r$  is a boundary of a strictly convex set) and, moreover,  $BD$  is tangent to  $M_r$  at  $B$  (See Fig. 5). To this aim note that  $\check{B}C \subset E_\Sigma$ , and hence for the energetic point  $B$  there is such a point  $Q_r \in M$  that  $|BQ_r| = r$  and  $B_r(Q_r) \cap \Sigma = \emptyset$ . The circle  $B_r(Q_r)$  touches  $M_r$  at the point  $B$  (in fact, the proof of Lemma 3.2 shows that  $(BQ_r)$  is perpendicular to the tangent line to  $M_r$  at point  $B$ ) and  $[BD]$  does not intersect  $B_r(Q_r)$  because

$$[BD] \cap B_r(Q_r) \subset \Sigma \cap B_r(Q_r) = \emptyset,$$

so  $[BD]$  is tangent to  $M_r$  at  $B$ , concluding the proof.  $\square$

*Remark 3.5.* If in the setting of Lemma 2.10  $M$  is a *strictly* convex curve then the claim of Lemma 2.10 is always true for local minimizers, i.e. the proof follows from the local arguments. In fact, the only part in the proof of Lemma where we use global minimality is showing that if  $(\check{B}C)$  is an arc in  $\Sigma \cap M_r$ , then  $C$  cannot be a branching point. We replace the respective part of the proof of Lemma 2.10 by the following argument. Suppose the contrary, i.e. that there is both a chord  $[CX] \subset \Sigma \cap N_r$  and a line segment  $[CY] \subset \Sigma \setminus N_r$ . If  $M$  is a *strictly* convex curve then all points of  $\text{Int}(\check{B}C)$  are energetic, and thus  $[CY]$  has to be a part of the tangent line to  $M_r$  at point  $C$ . Hence one can change  $\Sigma$  in the same way as at Fig 4: replace  $\Sigma \cap B_\varepsilon(C)$  by the Steiner tree connecting  $\Sigma \cap \partial B_\varepsilon(C)$  and a couple of segments of length  $o(\varepsilon^2)$  at the point  $C$  and  $D = \partial B_\varepsilon(C) \cap \check{B}C$  producing the competitor of strictly lower length.

*Proof of Lemma 2.12.* Claim (ii) is proven in [11]. To prove (i), suppose the contrary, i.e.  $\text{Int}(q_{S_1}) \cap \text{Int}(q_{S_2}) \neq \emptyset$  for  $S_1, S_2$  being some connected components of  $\Sigma \setminus N_r$  or arcs of  $\Sigma \cap M_r$ . Obviously, at least one of  $S_1, S_2$  is not an arc (it is  $S_1$  without loss of generality). Lemma 2.8 tells us that  $m(S_1) \leq 2$ . Obviously,  $m(S_1) \geq 1$ . If  $m(S_1) = 2$  one can cut the neighborhood of the corresponding energetic point. The length of  $\Sigma$  will decrease but it will still cover  $M$ . If  $m(S_1) = 1$  then Lemma 2.10 gives us a contradiction.  $\square$

### 3.1. Proof of the central Lemma.

*Proof of Lemma 2.17.* Obviously, if  $S$  is an arc then the compared values are equal.

Denote by  $Q_l$  and  $Q_r$  the ends of  $q_S$ . Let  $O$  be an intersection point of the perpendiculars to  $M$  in points  $Q_l$  and  $Q_r$ . Note that  $\text{wind}(q) = \angle Q_l O Q_r$  and denote it by  $\gamma$ . Note also that  $|Q_l O| \geq R$ ,  $|Q_r O| \geq R$ . Note that Lemmas 2.7 and 2.8 as well as Corollary 2.13 hold true when  $R > a_M(r) + r$  which is guaranteed when  $R > 5r$  (or  $R > 4.98r$  in the case when  $M$  is a circumference of radius  $R$ ), i.e. under the conditions of the statement being proven.

The following cases have to be considered.

- (1) Let  $S$  be a middle component. Then it has two entering points and one or two energetic points by Lemma 2.8. Since  $S$  is a Steiner connection for the set of its entering and energetic points, then there are four possible options of  $S$ .
  - (a) *The case  $n = 2, m = 2$ , and there exists a Steiner point adjacent to (adjacent to means that we consider  $S$  as a graph) both entering points (see Fig. 6). Note that if there exists a Steiner*

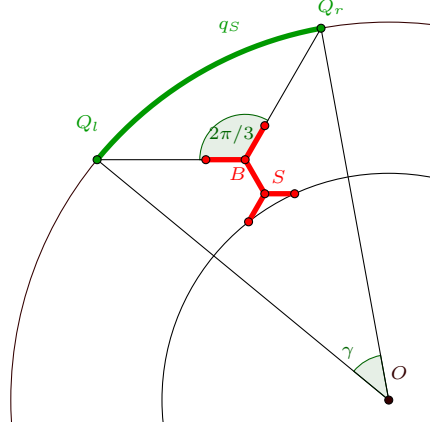


FIGURE 6. A general picture to the case 1a:  $n = 2, m = 2$ , there exists a Steiner point adjacent to both entering points.

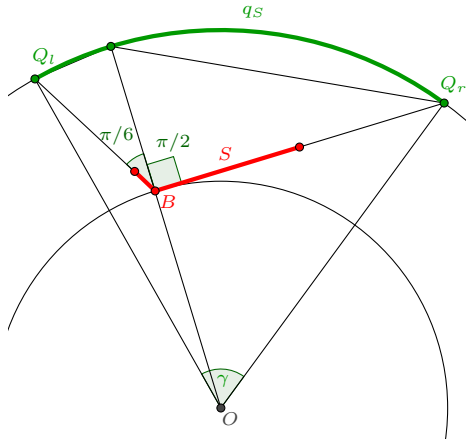


FIGURE 7. A marginal picture to the case 1a:  $n = 2, m = 2$ , there exists a Steiner point adjacent to both entering points.

point adjacent to both entering points then there exists a Steiner point (call it  $B$ ) adjacent to both energetic points. Clearly  $\text{wind}(S)$  is  $\pi/3$ . Let us prove that the covered arc  $q_S$  is less than  $\pi/3$ . We evaluate the arc bounded by continuations of segments starting from  $B$ . Easily this arc is the longest when  $B$  lies on  $M_r$  (it is a marginal case). Hence it is enough to look at the angle in  $N \setminus N_r$  of size  $2\pi/3$  with vertex  $B$  on  $M_r$ . It is well-known that the arc is the longest when  $S$  is tangent to  $M_r$  and when  $M$  is a circumference. In this case perpendicular line to  $M_r$  in point  $B$  splits the angle  $\angle Q_l B Q_r = 2\pi/3$  on angles size of  $\pi/2$  and  $\pi/6$  (see Fig. 7). In this case the size of arc is

$$\arccos\left(1 - \frac{1}{\alpha}\right) + \frac{\pi}{6} - \arcsin\left(\frac{1}{2}\left(1 - \frac{1}{\alpha}\right)\right),$$

where  $\alpha := R/r$ , hence it is less than  $\pi/3$  for  $\alpha \geq 2.9$ .

- (b) *The case  $n = 2$ ,  $m = 2$ , and there is no Steiner point adjacent to both entering points (see Fig. 8).* Denote the Steiner points of  $S$  by  $V_l$  and  $V_r$ . In this case  $\text{wind}(S) = \pi/3 + \pi/3 = 2\pi/3$ .

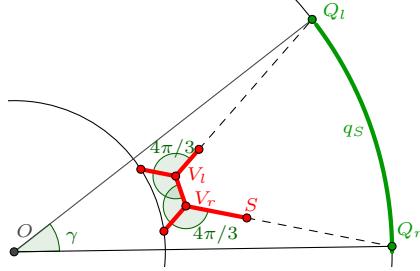


FIGURE 8. Picture to the case 1b:  
 $n = 2, m = 2$ , there is no Steiner point adjacent to both entering points.

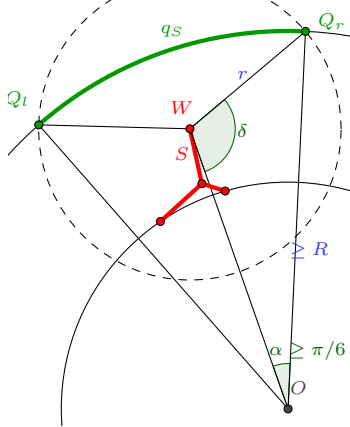


FIGURE 9. Picture to the case 1c:  
 $m = 2, n = 1$ .

Assume the contrary (it means that  $\gamma > 2\pi/3$ ) and connecting  $O$  with  $Q_l$  and  $Q_r$ , we get a (nonconvex) pentagon  $Q_l V_l V_r Q_r O$  with two angles equal to  $4\pi/3$  and one angle at least  $2\pi/3$ , which is impossible.

- (c) *The case  $n = 1$ ,  $m = 2$  and there is a Steiner point in  $S$  (see Fig. 9).* Clearly  $\text{wind}(S) = \pi/3$ . To prove the statement, assume the contrary (i.e.  $\gamma \geq \pi/3$ ) and as in the previous case connect  $O$  with  $Q_l$  and  $Q_r$ . Additionally connect  $O$  with the energetic point  $W$  so that the angle equal to  $\gamma$  splits into two parts; let us pick the largest one (without loss of generality it is  $\angle WOQ_r$ ).



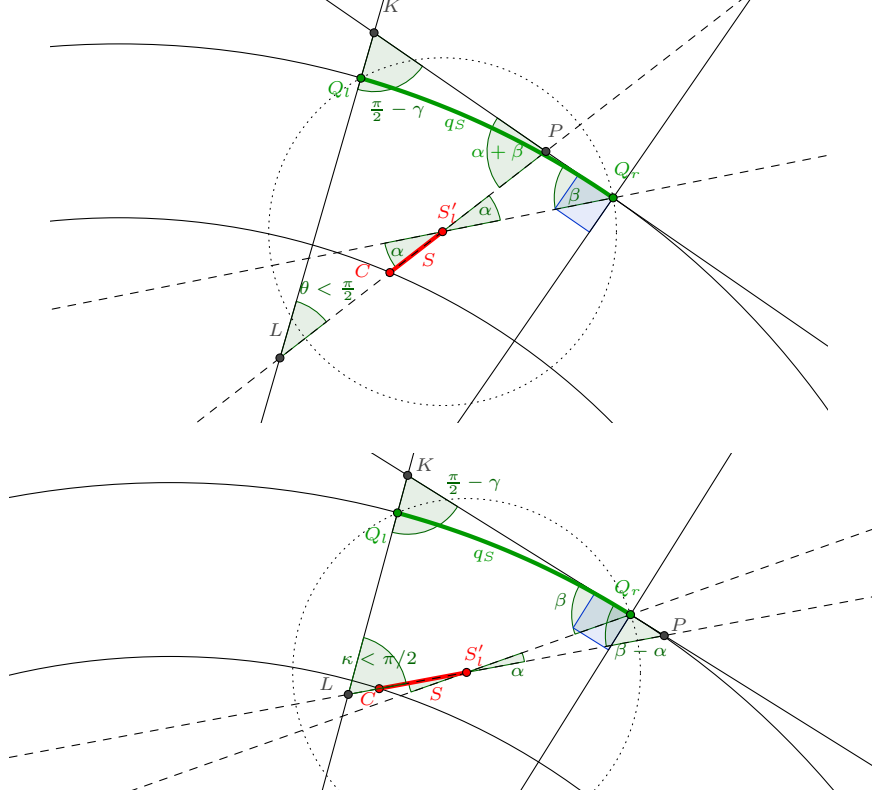


FIGURE 12. Picture to the case 2b: ending component,  $n = 1$ ,  $m = 1$ .

(2) Let  $S$  be an ending component (without loss of generality let it be the left one, so  $Q_r = A$ ). Recall that  $C$  denotes the branching point if  $S$  is a tripod and the entering point if  $S$  is a segment. Then there are two options:

- (a) The case  $n = 2$ ,  $m = 1$  (see Fig. 11). Note that  $S$  is a tripod:  $S = [BC] \cup [CW] \cup [CS'_l] \subset (N \setminus N_r)$ , where  $B \in M_r$ . Remark 3.4 implies that  $Q_r = [CS'_l] \cap M$  and  $Q_l = [CW] \cap M$  and  $|S'_l Q_r| = r = |WQ_l|$ ,  $B_r(Q_r) \cap \Sigma = B_r(Q_l) \cap \Sigma = \emptyset$ . Let  $K \in [OQ_l]$  be a point satisfying  $(Q_r K) \perp (OQ_r)$ . Then  $\alpha := \text{wind}(S) = \angle([BC], [CQ_r])$ ,  $\angle([CS'_l], [S'_l A]) = 0$  and  $\beta := \angle([CQ_r], [Q_r K]) = \angle([AS'_l], a)$ . We have to show  $\alpha + \beta > \gamma$ . Let  $P$  be the point of intersection of  $(KQ_r)$  and  $[BC]$ . Then  $\angle OKP = \pi/2 - \gamma$  and  $\angle KPC = \alpha + \beta$ . Assume the contrary i.e.  $\alpha + \beta \leq \gamma$ . Then  $\angle OKP + \angle KPC < \pi/2$  hence  $\angle KLP > \pi/2$ , where  $L$  is the point of intersection of  $(BC)$  and  $(OK)$ , but since  $\angle Q_l CL = 2\pi/3$ , then the sum of the angles of the triangle  $\Delta CLQ_l$  exceeds  $\pi$  which is impossible.
- (b) The case  $n = 1$ ,  $m = 1$  (see Fig. 12). In this case  $S = [CS'_l]$ , where  $C \in M_r$ ,  $|S'_l Q_r| = r$ , and  $\text{wind}(S) = 0$ . Denote by  $K$  such a point that  $K \in [OQ_l]$  and  $\angle OQ_r K = \pi/2$ . Define the points  $L := [S'_l C] \cap (OQ_l)$  and  $P := [BS'_l] \cap (Q_r K)$ , and introduce the angles  $\alpha := \angle PS'_l Q_r$  and  $\beta := \angle S'_l Q_r K$ .

The following two situations have to be considered. Note that  $|S'_l Q_l| = r$  otherwise one can replace  $[CS'_l] \cap B_\varepsilon(S'_l)$  on the part of the segment  $[DQ_r]$  where  $D = [CS'_l] \cap \partial B_\varepsilon(S'_l)$ .

- $\angle CS'_l Q_r \leq \pi$  (see Fig 12). Then  $\angle([AS'_l], a) = \beta$  and  $\angle([CS'_l], [S'_l A]) = \alpha$ , so that  $\text{wind}(S) + \angle([CS'_l], [S'_l A]) + \angle([AS'_l], a) = \alpha + \beta$ . Note that  $\angle S'_l PK = \alpha + \beta$  and  $\angle OKQ_r = \pi/2 - \gamma$ . If  $\alpha + \beta \leq \gamma$  (contrary to the claim being proven), then  $\angle OKP + \angle KPS'_l < \pi/2$  so  $\angle KLP > \pi/2$ , which is impossible because then  $|CQ_l| < |S'_l Q_l|$  which contradicts  $|S'_l Q_l| = r$ ,  $|CQ_l| \geq r$ .
- $\angle CS'_l Q_r > \pi$  (see Fig 12). In this case  $\angle([AS'_l], a) = \beta$  and  $\angle([CS'_l], [S'_l A]) = -\alpha$ , so that  $\text{wind}(S) + \angle([CS'_l], [S'_l A]) + \angle([AS'_l], a) = \beta - \alpha$  and we know that  $\angle KPC = \beta - \alpha$ .

If  $\beta - \alpha \leq \gamma$  (the contrary to the claim being proven), then  $\angle OKP + \angle KPC < \pi/2$ , which is impossible because then  $|CQ_l| < |S'_l Q_l|$  which contradicts  $|S'_l Q_l| = r$ ,  $|CQ_l| \geq r$ .  $\square$

*Proof of Corollary 2.4.* Let  $\Sigma$  be a local minimizer in the sense of Definition 2.3. We will show the claim with  $c := 1/4$ . Suppose it does not hold, i.e.  $\mathcal{H}^1(\Sigma) \leq \mathcal{H}^1(\Sigma_{\text{opt}}) + (R - 5r)/4$ . In view of Remark 2.9 Lemmas 2.7 and 2.8 as well as Corollary 2.13 remain true when  $r < R/5$ . Repeating line by line the proof of Theorem 2.2 without any change (we may do it because all the arguments used in this proof as well as in Lemma 2.17 are local, i.e. in fact one gets the contradiction with local minimality of  $\Sigma$  rather with the global one), we get that  $\Sigma$  is a horseshoe concluding the proof.  $\square$

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